Mechatronic Modeling and Domain Transformation of Multi-physics Systems

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Abstract: The enhanced definition of Mechatronics involves the four underlying characteristics of integrated, unified, unique, and systematic approaches. In this realm, Mechatronics is not limited to electro-mechanical systems, in the multi-physics sense, but involves other physical domains such as fluid and thermal. This paper summarizes the mechatronic approach to modeling. Linear graphs facilitate the development of state-space models of mechatronic systems, through this approach. The use of linear graphs in mechatronic modeling is outlined and an illustrative example of sound system modeling is given. Both time-domain and frequency-domain approaches are presented for the use of linear graphs. A mechatronic model of a multi-physics system may be simplified by converting all the physical domains into an equivalent single-domain system that is entirely in the output domain of the system. This approach of converting (transforming) physical domains is presented. An illustrative example of a pressure-controlled hydraulic actuator system that operates a mechanical load is given.

Keywords: Mechatronic Modeling, Multi-physics Systems, Integrated, Unified, Unique and Systematic Approach; Linear Graphs, Physical Domain Conversion/Transformation

1   Mechatronic Modeling

The field of Mechatronics concerns multi-physics systems, which typically have more than one physical domain (for example, domains among mechanical, electrical, fluid and thermal). The "enhanced" mechatronic approach constitutes the four characteristics[1]:

Integrated

Unified

Unique

Systematic.

Here, integrated means all physical domains in the system are considered together (i.e., concurrently or simultaneously). This is needed because, typically, there will be dynamic interactions among the physical domains.

Unified means, all physical domains are treated using similar (i.e., analogous) methodologies. No matter what the physical domain is, it will have two types of variables: through-variables and across-variables. Through-variables propagate unchanged through an element. Examples of through-variables are force, current, fluid flow rate, and heat transfer rate, which are all analogous. Across-variables are defined across a physical element, at one end (action end) with respect to the other end (reference end). Examples of across-variables are velocity, voltage, pressure, and temperature, which are all analogous. There will be two types of sources (inputs) in a system, as, through-type (T-type) sources whose independent variable is a through-variable, and across-type (A-type) sources, whose independent va-

riable is an across-variable. Examples of T-sources are force source, current source, fluid flow source, and heat transfer source. Examples of A-sources are velocity source, voltage source, pressure source, and temperature source. In view of these analogies, series-connected physical modules behave similarly across physical domains, and also parallel-connected physical modules behave similarly across physical domains. Approaches that exploit these analogies are indeed unified approaches.

Unique implies that at the end of the process (typically, modeling or design of a system), only a single result (a model or a design) is generated. Since an engineering system may have more than one model and more than one design, in order to yield a unique end result, the employed procedure needs to be "optimal" in some sense.

Systematic means, the underlying procedures need to be well-articulated and follow a clear set of steps. This means, there will not be any confusion as to what procedures (or program sequences) need to be followed in order to yield the end result. This also enables the software engineers to develop proper computer programs to carry out the underlying procedures.

The features of integrated, unified, unique, and systematic are necessarily possessed by proper "mechatronic" approaches of modeling or design.

2 Linear Graph Models

A model is a representation of the actual system. There are many types of models; in particular,

1. Physical Models (Prototypes)

2. Analytical Models

3. Computer (Numerical) Models (data tables, curves, programs, files, etc.)

4. Experimental Models (which use input/output experimental data from a real system, for "fitting" to a model; i.e., "model identification").

The focus of the present paper is analytical models. They are applicable in various practical situations[2-12].

Linear Graphs (LGs) provide an integrated and unified (multi-physics) tool to graphically represent a system model, which facilitate the development of a state-space model of an engineering system. An LG enables the visualization of the system structure, prior to the model formulation[1, 13-15]. It uses interconnected line segments (branches) to represent elements. It assists us in identifying similarities (in domain, structure, behavior, etc.) in the system. As the underlying methodology is "systematic" and graphical, LGs provide a basis for the development of computer-based modeling tools and software (e.g., MATLAB tools). Initially, the LG approach is applicable for lumped-parameter systems; yet it may be extended to distributed-parameter systems, through innovative extensions, which is beyond the scope of the present paper[14]. In the LG terminology, "linear" implies the use of "lines" to represent physical elements. Of course, the LG approach can be used to model nonlinear systems as well, which have nonlinear constitutive equations. Modeling considerations are important in the present context [1-12].

The main steps in formulating a lumped- parameter analytical model of a dynamic system are as follows:

1. Identify the system of interest (purpose, boundary),

2. Identify/specify the variables of interest (excitations/inputs, responses/outputs, etc.)

3. Approximate various segments (processes, phenomena) by ideal "lumped" elements, suitably interconnected. Draw a structural diagram for the system (e.g., linear graph), showing the structure (element/component interconnection) of the system,

4. Using the structural diagram,

(a) Write constitutive equations (physical laws) for elements (other than the "input elements" or "sources"),

(b) Write continuity (or conservation) equations for through-variables (those variable that do not change through an element) at junctions (nodes); E.g., equilibrium of forces at joints; current balance at nodes,

(c) Write compatibility (or loop) equations for across-variables (potential variables, path variables),

which are measured across an element, in closed paths (loops); E.g., for velocities—geometric connectivity; voltages—potential balance. *Note*: Compatibility is automatically satisfied in some systems, particularly because of the nature of the choice of across-variables for elements, for example, the reference end of an inertia (mass) element is the ground (inertial reference), which corresponds to zero velocity, similarly, the reference end of a source is the ground.

(d) Eliminate auxiliary (unwanted) variables, and

5. Express initial conditions using system variables (There are no boundary conditions for lumped models. This is because "time" is the only independent variable in the model).

In a sound system, the speaker unit will typically have a woofer, which is intended for the lower-frequency sound, and a tweeter, which is intended for the high-frequency sound. Consider the two electrical circuits, representing sound systems, shown in Fig.1 and Fig.2. In each circuit, the sound signal reaches the speaker, which is represented as an electrical resistor (with resistance R_1 or R_2), which may represent a woofer or a tweeter.

First, the expression of the electrical impedance (A-transfer function) of each circuit branch in Fig.1 and Fig.2 is given, in terms of the branch parameter (L_1, L_2, L, C, C_1, C_2, R_1, R_2) and the frequency ω of the sound signal (the input). Based on this information, it is determined which circuit corresponds to a woofer and which circuit corresponds to a tweeter.

First consider Fig.1. The electrical impedances of the circuit branches are: $L_1 j\omega$; $L_2 j\omega$; $\dfrac{1}{Cj\omega}$; and R_1.

It is seen that at low frequencies, the sound signal reaches the speaker, through the low-impedance path. At high frequencies, the sound signal is mostly blocked, since the sound source will encounter an overall high impedance. It follows that this circuit corresponds to a woofer.

Next consider Fig.2. The electrical impedances of the circuit branches are: $\dfrac{1}{C_1 j\omega}$; $\dfrac{1}{C_2 j\omega}$; $Lj\omega$; and R_2.

It is seen that at low frequencies, the sound signal is mostly blocked, since the sound source will encounter an overall high impedance. At high frequencies, the sound signal reaches the speaker, through the low-impedance path. It follows that this circuit corresponds to a tweeter.

Now consider the electrical circuit shown in Fig.3. It also represents the speaker circuit of a sound system, but in more detail. The speaker is represented by the resistor of resistance R_s. The effects of the inertia and the mechanical damping of the speaker are represented by an equivalent electrical capacitance C_m and an equivalent electrical resistance R_b, respectively. The electrical inductances L_s and L_c are the remaining circuit parameters.The circuit input is the sound signal of voltage $v_i(t)$ and the circuit output is the voltagevacross the speaker (of resistance R_s). Also, the current through the inductance L_s is i_s and the current through the inductance L_c is i_L. The voltage across the capacitor C_m is v_m (which corresponds to the velocity of the speaker).

First, a complete an oriented linear graph for the circuit is drawn. Using the linear graph, systematically, a complete state-space model is developed for the circuit. Using the state-space model, the input-output (I-O) differential equation for the circuit is derived.

The oriented linear graph of the circuit is shown in Fig.4. The particular choice of the primary loops (three), and all the nodes (four) are indicated on the LG.The system has three independent energy storage elements (L_s, L_c, C_m). Hence, the system is 3rd order, with three state variables, which are chosen according to the mechatronic approach, as state vector $\boldsymbol{x} = [i_s \quad i_L \quad v_m]^T$. Also, input vector $\boldsymbol{u} = [v_i(t)]$; and output vector $\boldsymbol{y} = [v]$.

Constitutive Equations (Physical Equations)

State-space Shell:

$$L_s: \quad L_s \frac{di_s}{dt} = v_s$$

$$L_c: \quad L_L \frac{di_L}{dt} = v_L$$

$$C_m: \quad C_m \frac{dv_m}{dt} = i_m$$

Remaining Constitutive Equations:

R_s: $v = R_s i$

R_b: $v_R = R_b i_R$

Independent Node Equations

Node 1: $i_i - i_s = 0$ {not useful because i_i is the dependent variable of the source}

Node 2: $i_s - i = 0$

Node 3: $i - i_L - i_R - i_m = 0$

Loop Equations

Loop 1: $-v_L - v - v_s + v_i(t) = 0$

Loop 2: $-v_m + v_L = 0$

Loop 3: $-v_m + v_R = 0$

Eliminate Auxiliary Variables:

$v_s = -v - v_L + v_i(t) = -R_s i - v_m + v_i(t)$
$= -R_s i_s - v_m + v_i(t)$

$v_L = v_m$

$i_m = i - i_L - i_R = i_s - i_L - \dfrac{v_R}{R_b} = i_s - i_L - \dfrac{1}{R_b} v_m$

State-space Model:

$$L_s \frac{di_s}{dt} = -R_s i_s - v_m + v_i(t) \qquad (1)$$

$$L_c \frac{di_L}{dt} = v_m \qquad (2)$$

$$C_m \frac{dv_m}{dt} = i_s - i_L - \frac{1}{R_b} v_m \qquad (3)$$

Output Equation: $v = R_s i = R_s i_s$

The overall state-space model is given by,

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Model Matrices:

$$A = \begin{bmatrix} -\dfrac{R_s}{L_s} & 0 & -\dfrac{1}{L_s} \\ 0 & 0 & \dfrac{1}{L_c} \\ \dfrac{1}{C_m} & -\dfrac{1}{C_m} & -\dfrac{1}{R_b C_m} \end{bmatrix}; \quad B = \begin{bmatrix} \dfrac{1}{L_s} \\ 0 \\ 0 \end{bmatrix};$$

$$C = \begin{bmatrix} R_s & 0 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Now, to obtain the I-O model, we need to get a single differential equation in terms of the input ($v_i(t)$) and the output ($R_s i_s$). Substitute (2) into (1) and (3), to eliminate v_m:

$$L_s \frac{di_s}{dt} = -R_s i_s - L_c \frac{di_L}{dt} + v_i(t) \qquad (4)$$

$$C_m L_c \frac{d^2 i_L}{dt^2} = i_s - i_L - \frac{L_c}{R_b} \frac{di_L}{dt} \qquad (5)$$

Differentiate (5):

$$C_m L_c \frac{d^3 i_L}{dt^3} = \frac{di_s}{dt} - \frac{di_L}{dt} - \frac{L_c}{R_b} \frac{d^2 i_L}{dt^2}$$

Substitute (4) in this, to eliminate i_L:

$$C_m L_c \frac{d^2}{dt^2} \left[-\frac{L_s}{L_c} \frac{di_s}{dt} - \frac{R_s}{L_c} i_s + \frac{1}{L_c} v_i(t) \right] =$$
$$\frac{di_s}{dt} - \left[-\frac{L_s}{L_c} \frac{di_s}{dt} - \frac{R_s}{L_c} i_s + \frac{1}{L_c} v_i(t) \right] -$$
$$\frac{L_c}{R_b} \frac{d}{dt} \left[-\frac{L_s}{L_c} \frac{di_s}{dt} - \frac{R_s}{L_c} i_s + \frac{1}{L_c} v_i(t) \right]$$

Or,

$$-C_m L_s \frac{d^3 i_s}{dt^3} - C_m R_s \frac{d^2 i_s}{dt^2} + C_m \frac{d^2 v_i(t)}{dt^2} =$$
$$\frac{di_s}{dt} + \frac{L_s}{L_c} \frac{di_s}{dt} + \frac{R_s}{L_c} i_s - \frac{1}{L_c} v_i(t) +$$
$$\frac{L_s}{R_b} \frac{d^2 i_s}{dt^2} + \frac{R_s}{R_b} \frac{di_s}{dt} - \frac{1}{R_b} \frac{dv_i(t)}{dt}$$

Multiply throughout by R_s:

$$-C_m L_s \frac{d^3 R_s i_s}{dt^3} - C_m R_s \frac{d^2 R_s i_s}{dt^2} + C_m R_s \frac{d^2 v_i(t)}{dt^2} =$$
$$\frac{dR_s i_s}{dt} + \frac{L_s}{L_c} \frac{dR_s i_s}{dt} + \frac{R_s}{L_c} R_s i_s - \frac{R_s}{L_c} v_i(t) +$$
$$\frac{L_s}{R_b} \frac{d^2 R_s i_s}{dt^2} + \frac{R_s}{R_b} \frac{dR_s i_s}{dt} - \frac{R_s}{R_b} \frac{dv_i(t)}{dt}$$

Substitute v:

$$-C_m L_s \frac{d^3 v}{dt^3} - C_m R_s \frac{d^2 v}{dt^2} + C_m R_s \frac{d^2 v_i(t)}{dt^2} =$$
$$\frac{dv}{dt} + \frac{L_s}{L_c} \frac{dv}{dt} + \frac{R_s}{L_c} v - \frac{R_s}{L_c} v_i(t) +$$
$$\frac{L_s}{R_b} \frac{d^2 v}{dt^2} + \frac{R_s}{R_b} \frac{dv}{dt} - \frac{R_s}{R_b} \frac{dv_i(t)}{dt}$$

Group the like terms, to give the I-O differential

equation:

$$C_m L_s \frac{d^3 v}{dt^3} + (C_m R_s + \frac{L_s}{R_b})\frac{d^2 v}{dt^2} + (1 + \frac{L_s}{L_c} + \frac{R_s}{R_b})\frac{dv}{dt} + \frac{R_s}{L_c}v =$$

$$C_m R_s \frac{d^2 v_i(t)}{dt^2} + \frac{R_s}{R_b}\frac{dv_i(t)}{dt} + \frac{R_s}{L_c}v_i(t)$$

Note: We should check now whether the units are consistent in all the terms in this equation (as a further check on the accuracy of the results). For this, it may be used that, in the Laplace domain, the terms $Ls, \frac{1}{Cs}, R$ all have the same units (Ohms, according to the constitutive equations of L, C, and R).

3 Transformation of Physical Domains

In the mechatronic modeling of a multi-physics system, once a model is obtained, further analysis can be facilitated by transforming all the physical domains in the model into the output domain. This will yield a single-domain equivalent model. To develop the associated methodology, it is adequate just consider a system that consists of only two physical domains. If the system has more than two physical domains, the domain transformation can be carried out by sequentially treating the output domain and just one other physical domain in the system.

To develop the domain transformation methodology, two types of coupling between two energy domains are available. They are transformer coupling and gyrator coupling. They are based on whether an across-variable or a through-variable in the input domain is relatedto an across-variable in the output domain. The underlying relations are presented next[16].

Transformer-coupled Systems

Suppose that the two domains of the system are coupled through a "generalized transformer." In the energy transfer port of a generalized transformer, the across-variable of the output domain is related to the across-variable of the input domain. First, the Thevenin theorem is applied to the input-domain subsystem, to determine the equivalent A-source $P_{oc}(s)$ and the equivalent generalized impedance Z_e in series, in the Thevenin equivalent LG. This result is shown in Fig.5(a). This is in the form of a transfer-function linear graph (TFLG).

Note: If the input subsystem is in the fluid domain, the equivalent Thevenin source $P_{oc}(s)$ is a pressure source, and Z_e is a fluid impedance (Pressure/Flow Rate). If the output domain is mechanical, f_1 is a force, v_1 is a velocity and Z-*subscript-e* is a mobilit. In any other domain, these quantities will take the corresponding meanings.

Now, it is needed to determine the equivalent A-source and the equivalent generalized impedance (in series) in the converted domain. The necessary steps of formulation for determining these are as follows:

1. Constitutive equation of impedance
2. Constitutive equations of transformer
3. Loop equation
4. Node equation
5. Carry out substitutions as needed.

This formulation is summarized in Table 1. The equivalent TFLG of the converted subsystem is shown in Fig.5(b).
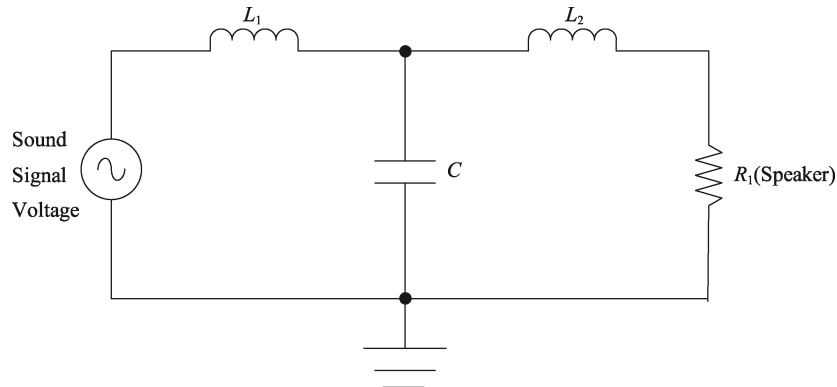

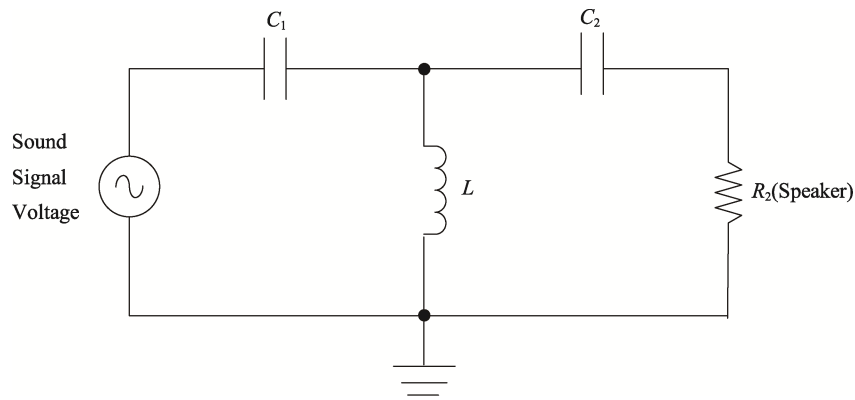**Fig.1   Electrical Circuit Representing a Speaker Unit of a Sound System**

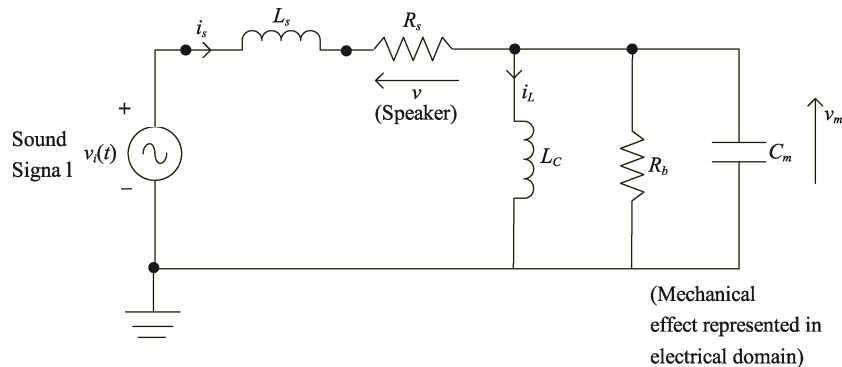**Fig.2   Another Electrical Circuit Representing a Speaker Unit of a Sound System**

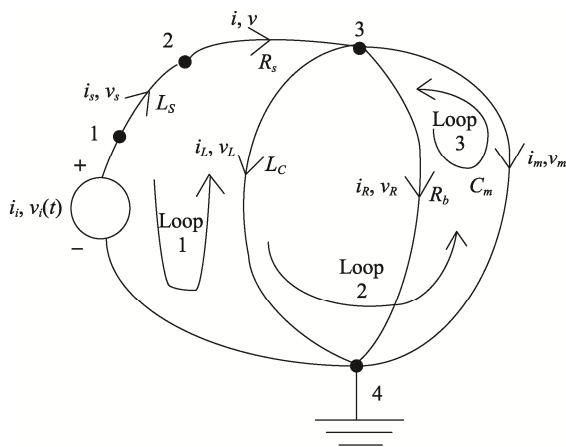**Fig.3   An Electrical Circuit Representing the Speaker Unit of a Sound System**

Fig.4   The Oriented Linear Graph of the Circuit

Gyrator-coupled Systems

Suppose that the two domains of the system are coupled through a "generalized gyrator." In its energy transfer port, the across-variable of the output domain is related to the through-variable of the input domain.

As before, the Thevenin theorem is applied to the input-domain subsystem, to determine the equivalent A-source $P_{oc}(s)$ and the equivalent generalized impedance Z_e in series, in the Thevenin equivalent LG. This result is shown in Fig.6(a). This is in the form of a transfer-function linear graph (TFLG).

Finally, it is required to determine the equivalent A-source and the equivalent generalized impedance (in series) in the converted domain. The necessary steps of formulation for determining these are as follows:

1. Constitutive equation of impedance
2. Constitutive equations of transformer
3. Loop equation
4. Node equation
5. Carry out substitutions as needed.

This formulation is summarized in Table 2. The equivalent TFLG of the converted subsystem is shown in Fig.6(b).
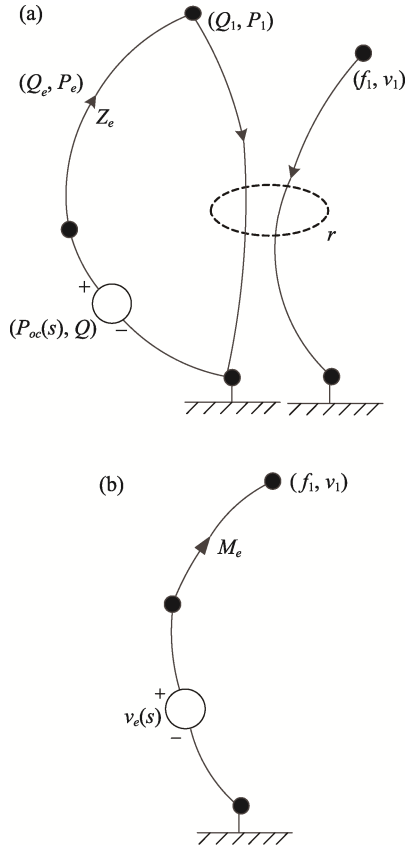
Fig.5   Transformer-coupled System. (a) Thevenin equivalent LG of the subsystem to be converted; (b) Equivalent transfer-function LG of the subsystem in the new domain
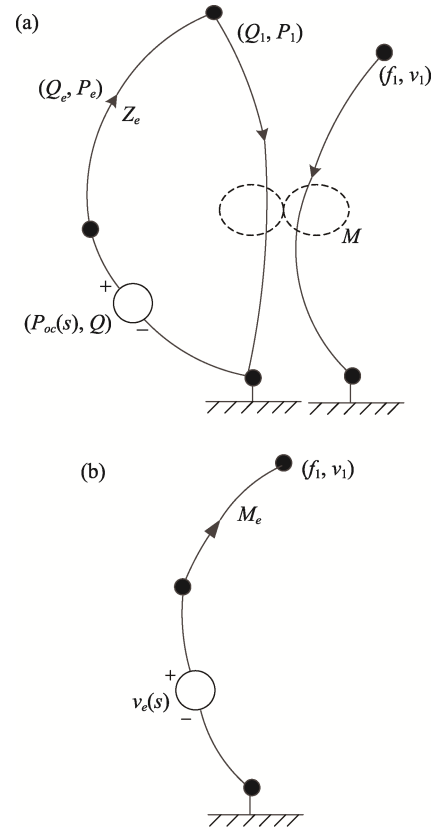

Fig.6   Gyrator-coupled System. (a) Thevenin equivalent LG of the subsystem to be converted; (b) Equivalent transfer-function LG of the subsystem in the new domain

As an illustrative example for the mechatronic modeling of a multi-physics system and converting it into a single domain (the output domain), consider the system consisting of both mechanical components and fluid components, as shown in Fig.7. In this system, a pump of pressure $P_s(t)$, which is a pressure source (the system input), pumps water through a uniform horizontal thin pipe, into a horizontal cylinder of area of cross-section A and a leak-proof piston, which serve as the hydraulic actuator that drives a mechanical load. The combined mass of the actuator piston and the mechanical load is m, the resisting stiffness of the mechanical load is k, and the combined viscous damping constant of the actuator piston and the mechanical load is b. The fluid inertance in the pipe is represented by I and the fluid resistance in the pipe is represented by R. The pressure ripples in the water

flow of the pipe are suppressed before the flow enters the actuator, by means of an energy absorber consisting of an open tank of fluid capacitance C_f (assumed to be constant). The water in this buffer tank goes through a valve of fluid resistance R_v before entering the actuator cylinder.

Note: Even though the fluid resistance R_v of the valve is adjustable (by operating the valve), it is assumed to be a constant (i.e., the valve opening is kept the same throughout the operation).

The velocity v_m of the load m (also, of the actuator piston) is the system output.

First, a complete linear graph (LG) is drawn for the system. Using this linear graph, a complete state space model is developed for the system. From the state-space model, the input-output differential equation is determined (input $= P_s(t)$, output $= v_m$). From

Table 1   Domain Conversion of a Transformer-coupled System

Formulation Step	Result
Equivalent-impedance Constitutive Equation	$P_e = Z_e Q_e$
Transformer Constitutive Equations	$v_1 = rP_1$; $f_1 = -\dfrac{1}{r}Q_1$; r = Transformer Parameter
Loop Equation	$-P_1 - P_e + P_{oc}(s) = 0$
Node Equation	$Q_e - Q_1 = 0$
Result after Substitutions	$v_1 = rP_{oc}(s) - rZ_e Q_1$
Final Result	$v_1 = rP_{oc}(s) + r^2 Z_e f_1$ or $v_1 = V_e(s) + M_e f_1$
Converted Equivalent A-source	$V_e(s) = rP_{oc}(s)$
Converted Equivalent Generalized Impedance (In Series)	$M_e = r^2 Z_e$

Table 2   Domain Conversion of a Gyrator-coupled System

| Formulation Step	Result
Equivalent-impedance Constitutive Equation	$P_e = Z_e Q_e$
Gyrator Constitutive Equations	$v_1 = MQ_1$; $f_1 = -\dfrac{1}{M}P_1$; M = Gyrator Parameter
Loop Equation	$-P_1 - P_e + P_{oc}(s) = 0$
Node Equation	$Q_e - Q_1 = 0$
Result after Substitutions	$v_1 = \dfrac{M}{Z_e}[P_{oc}(s) - P_1]$
Final Result	$v_1 = \dfrac{M}{Z_e}P_{oc}(s) + \dfrac{M^2}{Z_e}f_1$ or $v_1 = V_e(s) + M_e f_1$
Converted Equivalent A-source	$V_e(s) = \dfrac{M}{Z_e}P_{oc}(s)$
Converted Equivalent Generalized Impedance (In Series)	$M_e = \dfrac{M^2}{Z_e}$
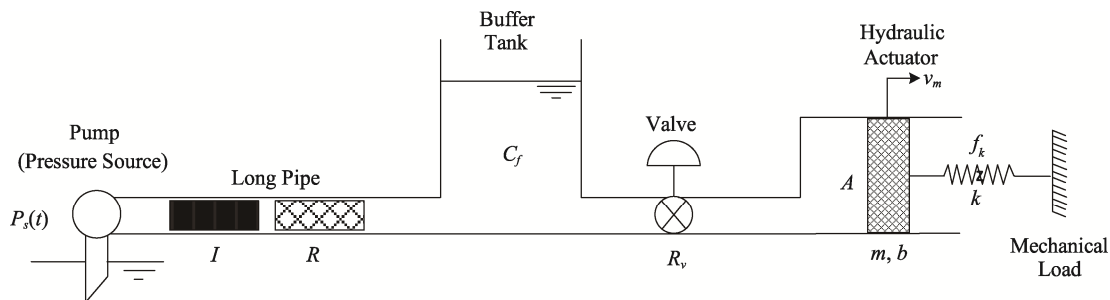
Fig.7   A Hydraulic Actuator System for Operating a Mechanical Load

that equation, the system transfer function is obtained.

In the second stage of this example, the transfer-function linear graph (TFLG) corresponding to the original LG is sketched. It is appropriately reduced by combining branches. It is then converted it into a TFLG that is entirely in the mechanical domain (the output domain). From that, the system transfer function is obtained while confirming that it is identical to

what was obtained before.

The linear graph of the system is shown in Fig.8.The particular choice of the primary loops (five), and the primary nodes (6 – 1 = 5) are indicated on the LG. The system has four independent energy storage elements (I, C_f, m, k). Hence, the system is 4th order, with four state variables. The state variables are chosen according to the systematic approach, as:

Q_I = volume flow rate of the water in the pipe before reaching the buffer tank,

P_f = gauge pressure of the water at the bottom of the buffer tank,

v_m = velocity of the mechanical load (and also of the actuator piston), and

f_k = spring force of the mechanical load (attached to the actuator piston).

State-space Shell:

$$I\frac{dQ_I}{dt} = P_I$$

$$C_f\frac{dP_f}{dt} = Q_f$$

$$m\frac{dv_m}{dt} = f_m$$

$$\frac{df_k}{dt} = kv_k$$

Other Constitutive Equations:

$$P_R = RQ_R$$
$$P_v = R_vQ_v$$
$$f_b = bv_b$$
$$\left.\begin{array}{l} v_2 = \dfrac{Q_1}{A} \\ f_2 = -AP_1 \end{array}\right\} \text{Fluid-mechanical gyrator}$$

Node Equations:

Node 1: $Q_s - Q_I = 0$ (useless); Node 2: $Q_I - Q_R = 0$;
Node 3: $Q_R - Q_f - Q_v = 0$;
Node 4: $Q_v - Q_1 = 0$; Node 5: $-f_2 - f_m - f_k - f_b = 0$

Loop Equations:

Loop 1: $-P_f - P_R - P_I + P_s(t) = 0$; Loop 2:
$-P_1 - P_v + P_f = 0$; Loop3: $-v_m + v_2 = 0$;
Loop 4: $-v_k + v_m = 0$; Loop 5: $-v_b + v_m = 0$

Eliminate Auxiliary Variables:

$$P_I = -P_f - P_R + P_s(t) = -P_f - RQ_R + P_s(t)$$
$$= -P_f - RQ_I + P_s(t)$$

{From Loop1, R-constitutive, and Node 2 equations}

$$Q_f = Q_R - Q_v = Q_I - Q_1 = Q_I - Av_2 = Q_I - Av_m$$

{From Node 3, Node 2, Node 4, and Loop 3 equations}

$$f_m = -f_2 - f_k - f_b = AP_1 - f_k - bv_b$$
$$= A(-P_v + P_f) - f_k - bv_m$$
$$= A(-R_vQ_v + P_f) - f_k - bv_m$$
$$= -AR_vQ_1 + AP_f - f_k - bv_m$$
$$= -AR_vAv_2 + AP_f - f_k - bv_m$$
$$= -A^2R_vv_m + AP_f - f_k - bv_m$$
$$= -(A^2R_v + b)v_m + AP_f - f_k$$

{From Node 5, Gyrator-constitutive, Loop 2, Loop 5, R_v-constitutive, Node 4, and Loop 3 equations}

$$v_k = v_m \quad \text{\{From Loop 4 equation\}}$$

State Equations:

$$I\frac{dQ_I}{dt} = -RQ_I - P_f + P_s(t)$$

$$C_f\frac{dP_f}{dt} = Q_I - Av_m$$

$$m\frac{dv_m}{dt} = AP_f - (A^2R_v + b)v_m - f_k$$

$$\frac{df_k}{dt} = kv_m$$

Output = v_m

Vector-matrix Formulation:

State vector $\mathbf{x} = \begin{bmatrix} Q_I & P_f & v_m & f_k \end{bmatrix}^T$

Input vector $\mathbf{u} = [u] = [P_s(t)]$

Output vector $\mathbf{y} = [y] = [v_m]$

State-space Model:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

with

$$A = \begin{bmatrix} -\dfrac{R}{I} & -\dfrac{1}{I} & 0 & 0 \\ \dfrac{1}{C_f} & 0 & -\dfrac{A}{C_f} & 0 \\ 0 & \dfrac{A}{m} & -\dfrac{(A^2 R_v + b)}{m} & -\dfrac{1}{m} \\ 0 & 0 & k & 0 \end{bmatrix} ;$$

$$B = \begin{bmatrix} \dfrac{1}{I} \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} ; D = \begin{bmatrix} 0 \end{bmatrix}$$

To get the input-output differential equations, first the state equations are written as:

$$\dot{x}_1 = -a_{11}x_1 - a_{12}x_2 + b_1 u \qquad (6)$$

$$\dot{x}_2 = a_{21}x_1 - a_{23}y \qquad (7)$$

$$\dot{y} = a_{32}x_2 - a_{33}y - a_{34}x_4 \qquad (8)$$

$$\dot{x}_4 = a_{43}y \qquad (9)$$

with,

$$a_{11} = \frac{R}{I}; \ a_{12} = \frac{1}{I}; \ a_{21} = \frac{1}{C_f}; \ a_{23} = \frac{A}{C_f}; \ a_{32} = \frac{A}{m};$$

$$a_{33} = \frac{(A^2 R_v + b)}{m}; \ a_{34} = \frac{1}{m}; \ a_{43} = k; \ b_1 = \frac{1}{I}$$

Now, these 4 state equations have to be combined into a single differential equation in u and y only. This is done as follows.

Substitute (8) into (9) to eliminate x_4:

$$m\frac{d}{dt}\left[-\dot{y} + a_{32}x_2 - a_{33}y\right] = a_{43}y \rightarrow$$

$$m\left[\ddot{y} + a_{33}\dot{y}\right] + ky = A\dot{x}_2 \qquad (10)$$

Substitute (7) into (6) to eliminate x_1:

$$C_f \frac{d}{dt}\left[\dot{x}_2 + a_{23}y\right] = -a_{11}C_f\left[\dot{x}_2 + a_{23}y\right] - a_{12}x_2 + b_1 u \rightarrow$$

$$IC_f \ddot{x}_2 + RC_f \dot{x}_2 + x_2 = -AI\dot{y} - ARy + u \qquad (11)$$

Substitute (10) into differentiated (11) to eliminate x_2:

$$\frac{IC_f}{A}\frac{d^2}{dt^2}\left[m\ddot{y} + ma_{33}\dot{y} + ky\right] +$$

$$\frac{RC_f}{A}\frac{d}{dt}\left[m\ddot{y} + ma_{33}\dot{y} + ky\right] + \qquad \rightarrow$$

$$\frac{1}{A}\left[m\ddot{y} + ma_{33}\dot{y} + ky\right] = -AI\ddot{y} - AR\dot{y} + \dot{u}$$

$$\frac{IC_f}{A}[m\frac{d^4 y}{dt^4} + ma_{33}\frac{d^3 y}{dt^3} + k\frac{d^2 y}{dt^2}] +$$

$$\frac{RC_f}{A}[m\frac{d^3 y}{dt^3} + ma_{33}\frac{d^2 y}{dt^2} + k\frac{dy}{dt}] +$$

$$\frac{1}{A}[m\frac{d^2 y}{dt^2} + ma_{33}\frac{dy}{dt} + ky] =$$

$$-AI\frac{d^2 y}{dt^2} - AR\frac{dy}{dt} + \frac{du}{dt}$$

By substituting for a_{33} and rearranging, the I-O differential equation is obtained:

$$\frac{IC_f}{A}[m\frac{d^4 y}{dt^4} + (A^2 R_v + b)\frac{d^3 y}{dt^3} + k\frac{d^2 y}{dt^2}] +$$

$$\frac{RC_f}{A}[m\frac{d^3 y}{dt^3} + (A^2 R_v + b)\frac{d^2 y}{dt^2} + k\frac{dy}{dt}] +$$

$$\frac{1}{A}[m\frac{d^2 y}{dt^2} + (A^2 R_v + b)\frac{dy}{dt} + ky] = \qquad \rightarrow$$

$$-AI\frac{d^2 y}{dt^2} - AR\frac{dy}{dt} + \frac{du}{dt}$$

$$\frac{IC_f m}{A}\frac{d^4 y}{dt^4} + \frac{IC_f(A^2 R_v + b)}{A}\frac{d^3 y}{dt^3} + \frac{IC_f k}{A}\frac{d^2 y}{dt^2} +$$

$$\frac{RC_f m}{A}\frac{d^3 y}{dt^3} + \frac{RC_f(A^2 R_v + b)}{A}\frac{d^2 y}{dt^2} + \frac{RC_f k}{A}\frac{dy}{dt} +$$

$$\frac{m}{A}\frac{d^2 y}{dt^2} + \frac{(A^2 R_v + b)}{A}\frac{dy}{dt} + \frac{k}{A}y =$$

$$-AI\frac{d^2 y}{dt^2} - AR\frac{dy}{dt} + \frac{du}{dt}$$

Now, group the like terms, and multiply throughout by A:

$$IC_f m\frac{d^4 y}{dt^4} + \left[IC_f(A^2 R_v + b) + RC_f m\right]\frac{d^3 y}{dt^3} +$$

$$\left[IC_f k + RC_f(A^2 R_v + b) + m + A^2 I\right]\frac{d^2 y}{dt^2} +$$

$$\left[RC_f k + A^2(R_v + R) + b\right]\frac{dy}{dt} + ky = A\frac{du}{dt}$$

Or,

$$IC_f m\frac{d^4 v_m}{dt^4} + \left[IC_f(A^2 R_v + b) + RC_f m\right]\frac{d^3 v_m}{dt^3} +$$

$$\left[IC_f k + RC_f(A^2 R_v + b) + m + A^2 I\right]\frac{d^2 v_m}{dt^2} +$$

$$\left[RC_f k + A^2(R_v + R) + b\right]\frac{dv_m}{dt} + kv_m = A\frac{dP_s(s)}{dt}$$

As expected, a 4th order input-output differential

equation model is obtained for this 4th order system (which has 4 independent energy storage elements). The corresponding system transfer function (by changing $\dfrac{d}{dt}$ to s):

$$\frac{v_m(s)}{P_s(s)} = \frac{As}{\begin{aligned} & IC_f m s^4 + \left[IC_f(A^2 R_v + b) + RC_f m \right] s^3 + \\ & \left[IC_f k + RC_f(A^2 R_v + b) + m + A^2 I \right] s^2 + \\ & \left[RC_f k + A^2(R_v + R) + b \right] s + k \end{aligned}} \quad (12)$$

For the next part of the example, the transfer-function linear graph (TFLG) of the LG in Fig.8 is sketched as in Fig.9. The fluid domain is now converted into the Thevenin form, as shown in Fig.10. Further reduction of the LG has been done as well, by combining the two parallel branches: the k-branch and the b-branch in the mechanical domain, into a single branch with mobility M. Here,

$$M = \frac{\dfrac{s}{k} \times \dfrac{1}{b}}{\left(\dfrac{s}{k} + \dfrac{1}{b} \right)} = \frac{s}{bs + k} \quad (13)$$

The following results are obtained by following the usual procedure for Thevenin circuit development:

Equivalent (open-circuit) pressure source,

$$P_{oc}(s) = \frac{1}{\left(Is + R + \dfrac{1}{C_f s} \right)} \times \frac{1}{C_f s} P_s(s) = \frac{1}{C_f s(Is + R) + 1} P_s(s) \quad (14)$$

Note: The potential divider (pressure divider, in the fluid domain) method is used in writing this equation.

Equivalent fluid impedance,

$$Z_e = \frac{(Is + R) \times \dfrac{1}{C_f s}}{(Is + R) + \dfrac{1}{C_f s}} + R_v = \frac{Is + R}{C_f s(Is + R) + 1} + R_v \quad (15)$$

Note: Here, after killing (i.e., shorting) the fluid source, combine the resulting two parallel branches and then add the series branch.

Next, convert the fluid domain into an equivalent mechanical domain, through the gyrator (fluid-mechanical energy converter). The equivalent TFLG shown in Fig.11 is obtained. This equivalent system is entirely in the mechanical domain.

The domain conversion of the original gyrator-coupled two-domain system is carried out as follows:

Constitutive Equations for Gyrator:

$$v_m = \frac{1}{A} Q_1; \; f_2 = -AP_1$$

Loop Equation (with constitution equation for Z_e substituted): $-P_1 - Z_e Q_1 + P_{oc}(s) = 0$

Substitute the node equation $Q_1 - Q_1 = 0$:

$$-P_1 - Z_e Q_1 + P_{oc}(s) = 0$$

Substitute the Gyrator equations, to eliminate the fluid-domain variables:

$$\frac{f_2}{A} - Z_e A v_m + P_{oc}(s) = 0,$$

or: $v_m = \dfrac{1}{AZ_e} P_{oc}(s) + \dfrac{1}{A^2 Z_e} f_2$

This result has only the mechanical side variables of the output branch of the gyrator (i.e., v_m and f_2).

From this result, it is obtained:

Equivalent velocity source,

$$v_e(s) = \frac{1}{AZ_e} P_{oc}(s) \quad (16)$$

Equivalent series mobility, $M_e = \dfrac{1}{A^2 Z_e}$ $\quad (17)$

In Fig.11, the equation of the first branch in the equivalent system is:

$$v_m = v_e(s) - M_e f_2 \quad (18)$$

Note: The direction of f_2 in the first branch of the TFLG that is entirely in the mechanical domain (see Fig.11) is opposite of what is in the output branch of the gyrator in the original TFLG (see Fig.10). This is the reason for the sign reversal of the f_2 term in (18).

The combined mobility of the two parallel branches in Fig.11 is:

$$M_m = \frac{M \times \frac{1}{ms}}{(M + \frac{1}{ms})} = \frac{M}{Mms+1} \qquad (19)$$

To get the system transfer function, an expression for v_m (the output) is needed. For this, use the familiar, "potential divider" approach (in the present "mechanical" situation, use the analogous, "velocity divider" approach) in Fig.11: $v_m = \frac{M_m}{(M_e + M_m)} \times v_e(s)$

Into this equation, substitute (16), then (14):

$$v_m = \frac{M_m}{(M_e + M_m)} \times \frac{1}{AZ_e} P_{oc}(s) = \frac{1}{AZ_e(M_e/M_m+1)} \times \frac{1}{\left[C_f s(Is+R)+1\right]} P_s(s)$$

The system transfer function is,

$$TF = \frac{v_m}{P_s(s)} = \frac{1}{AZ_e(M_e/M_m+1)\left[C_f s(Is+R)+1\right]}$$

Now substitute (19), (17), and (13):

$$TF = \frac{1}{AZ_e(\frac{1}{A^2 Z_e} \times \frac{Mms+1}{M}+1)\left[C_f s(Is+R)+1\right]}$$

$$= \frac{A}{(ms+\frac{1}{M}+A^2 Z_e)\left[C_f s(Is+R)+1\right]}$$

$$= \frac{A}{(ms+\frac{bs+k}{s}+A^2 Z_e)\left[C_f s(Is+R)+1\right]}$$

Now substitute (15):

$$TF = \frac{A}{\left[ms+\frac{bs+k}{s}+A^2\left\{\frac{Is+R}{C_f s(Is+R)+1}+R_v\right\}\right]\left[C_f s(Is+R)+1\right]}$$

$$= \frac{As}{\left[ms^2+bs+k+A^2 s\left\{\frac{Is+R}{C_f s(Is+R)+1}+R_v\right\}\right]\left[C_f s(Is+R)+1\right]}$$

$$= \frac{As}{\left[(ms^2+bs+k)\left\{C_f s(Is+R)+1\right\}+A^2 s(Is+R)+A^2 R_v s\left\{C_f s(Is+R)+1\right\}\right]}$$

Simplify:

$$TF = \frac{As}{\begin{bmatrix}(ms^2+bs+k)\left\{C_f s(Is+R)+1\right\}+\\ A^2 Is^2+A^2 Rs+A^2 R_v C_f Is^3+\\ A^2 R_v C_f Rs^2+A^2 R_v s\end{bmatrix}}$$

Or,

$$TF = \frac{As}{\begin{bmatrix}(ms^2 C_f s(Is+R)+bsC_f s(Is+R)+\\ kC_f s(Is+R)+ms^2+bs+k+A^2 Is^2+\\ A^2 Rs+A^2 R_v C_f Is^3+A^2 R_v C_f Rs^2+A^2 R_v s\end{bmatrix}}$$

Or,

$$TF = \frac{As}{\begin{bmatrix}mC_f Is^4+mC_f Rs^3+bC_f Is^3+RbC_f s^2+\\ kC_f Is^2+RkC_f s+ms^2+bs+k+A^2 Is^2+\\ A^2 Rs+A^2 R_v C_f Is^3+A^2 R_v C_f Rs^2+A^2 R_v s\end{bmatrix}}$$

Or,

$$TF = \frac{As}{\begin{aligned}&mC_f Is^4+\left[A^2 R_v C_f I+bC_f I+mC_f R\right]s^3+\\ &\left[kC_f I+A^2 R_v C_f R+RbC_f+m+A^2 I\right]s^2+\\ &\left[RkC_f+A^2 R+A^2 R_v+b\right]s+k\end{aligned}}$$

This result is identical to (12), which is what was obtained from the time domain approach.

4 Conclusion

Mechatronic modeling incorporates the four features, integrated, unified, unique, and systematic. This definition extends to mechatronic design and control as well. The linear graphs (LGs) facilitate the methodology of mechatronic modeling. The time-domain modeling by LGs can be extended into frequency-domain modeling. This paper outlined the methodology of mechatronic modeling in both time domain and frequency domain, through the use of linear graphs. A typical mechatronic system incorporates more than one physical domain. The model of such a multi-physics system can be simplified for the subsequent analysis by transforming it into an equivalent single-domain model, which is entirely in the output domain. The paper described this method of domain
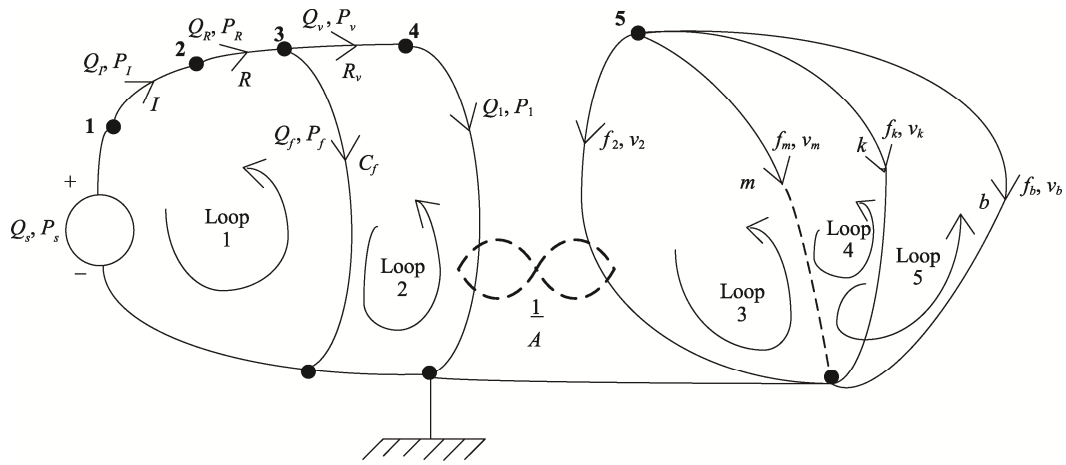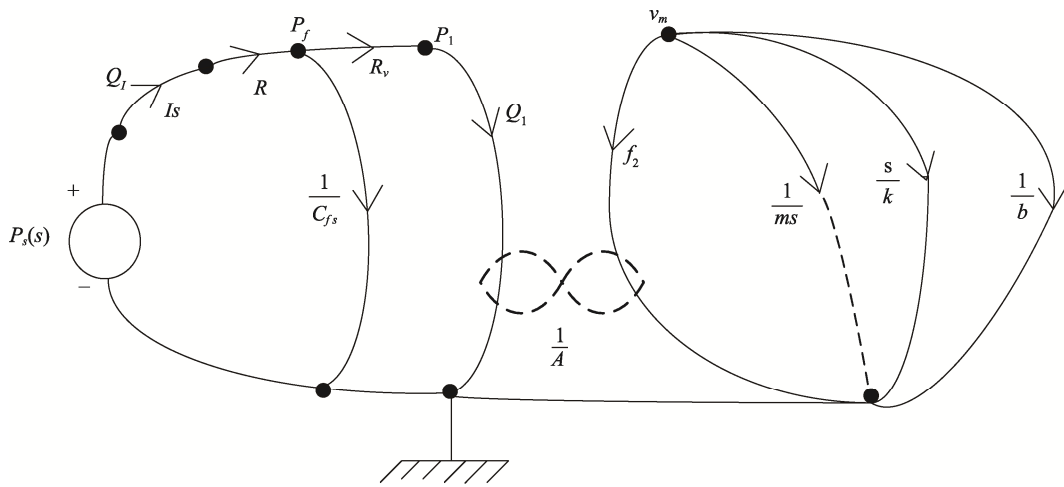
Fig.8   Linear Graph of the System

Fig.9   The Transfer-function Linear Graph of the Multi-domain System (Fluid-mechanical)
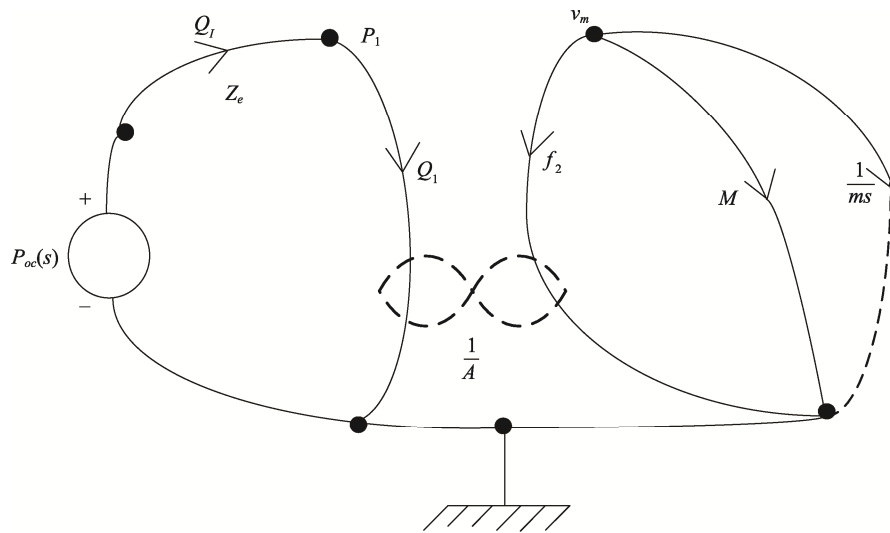

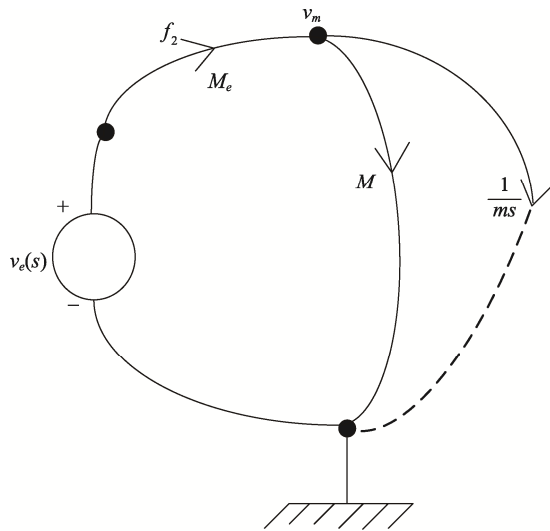**Fig.10   The TFLG with the Fluid Domain in the Thevenin Form**

Fig.11 The Equivalent TFLG Entirely in the Mechanical Domain

transformation (conversion). Practical engineering examples were presented to illustrate the application of the presented methodologies.

Acknowledgment

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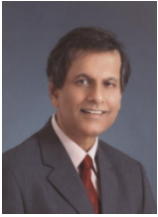
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