Minimal Realization of Linear Graph Models for Multi-physics Systems

Clarence W. DE SILVA

(*Department of Mechanical Engineering, The University of British Columbia, Vancouver, BC V6T 1Z4*)

Abstract: An engineering system may consist of several different types of components, belonging to such physical "domains" as mechanical, electrical, fluid, and thermal. It is termed a multi-domain (or multi-physics) system. The present paper concerns the use of linear graphs (LGs) to generate a minimal model for a multi-physics system. A state-space model has to be a minimal realization. Specifically, the number of state variables in the model should be the minimum number that can completely represent the dynamic state of the system. This choice is not straightforward. Initially, state variables are assigned to all the energy-storage elements of the system. However, some of the energy storage elements may not be independent, and then some of the chosen state variables will be redundant. An approach is presented in the paper, with illustrative examples in the mixed fluid-mechanical domains, to illustrate a way to recognize dependent energy storage elements and thereby obtain a minimal state-space model. System analysis in the frequency domain is known to be more convenient than in the time domain, mainly because the relevant operations are algebraic rather than differential. For achieving this objective, the state space model has to be converted into a transfer function. The direct way is to first convert the state-space model into the input-output differential equation, and then substitute the time derivative by the Laplace variable. This approach is shown in the paper. The same result can be obtained through the transfer function linear graph (TF LG) of the system. In a multi-physics system, first the physical domains have to be converted into an equivalent single domain (preferably, the output domain of the system), when using the method of TFLG. This procedure is illustrated as well, in the present paper.

Key words: Multi-physics Modelling, Mechatronic Systems, Linear Graphs, Dependent Energy Storage Elements, Redundant State Variables, Minimal State-space Realization, Domain Conversion, Equivalent Models, Frequency-domain Model.

1 Introduction

An engineering dynamic system, which is the focus of the present paper, may consist of several different physical types of components, belonging to such physical "domains" as mechanical, electrical, fluid, and thermal. It is termed a multi-domain (or multi-physics) system. Modeling of dynamic systems is applicable in a wide range of disciplines and applications including sociology [1], education [2], biomedical engineering [3], process engineering [4-5], electrical power systems [6], and robotics [7]. There are many approaches for modelling and engineering dynamic system. They include linear graphs[8,9], bond graphs [9], state-machine models or discrete-system models [10], logic-based models [11], empirical models [12], and so on [13].

A systematic, unified, and integrated approach is desirable for developing an analytical model, particularly a state-space model, of an engineering dynamic system [9]. The present paper uses the approach of linear graphs for modeling a multi-physics system in this manner [9, 14, 15]. This can be done in both the time domain and the frequency domain. Modeling in the frequency domain is particularly convenient and useful. First, the linear graph (LG) is converted into a multi-domain transfer-function linear graphs (TF LGs). Then, it is converted into an equivalent single domain TF LG, which is analyzed to obtain the transfer-function model.

A state-space model has to be a minimal realization. Specifically, the number of state variables used in the model should be the minimum number that can completely represent the dynamic state of the system. This choice is not straightforward. Initially, state variables are assigned to all the energy-storage elements of the system. However, some of the energy storage elements in the system may not be independent, and then some of the chosen state variables will be redundant. It is not always easy to know which energy storage elements are dependent, and then, we may end of with redundant state variables. Several ways exist to recognize the presence of dependent energy storage elements and thereby avoid redundant state variables. One approach is to use the concepts of graph tree [9]. Another approach is presented in the present paper, through illustrative examples in the mixed fluid-mechanical domains.

System analysis in the frequency domain is known to be more convenient than in the time domain, mainly because the relevant operations are algebraic rather than differential. For achieving this objective, the state space model has to be converted in a transfer function. The direct way is to first convert the state-space model into the corresponding input-output differential equation, and then substitute the time derivative by the Laplace variable. This approach is shown first. The same result can be obtained through the transfer function linear graph (TFLG) of the system. When using the method of TFLG in a multi-physics system, first the physical domains have to be converted into an equivalent single domain (preferably, the output domain of the system). This procedure is illustrated as well, in the present paper.

2 Series-connected Capacitors

If two components are connected in series, it is known that their through-variables are common (i.e., equal) and the across-variables add. A physically-realizable fluid-mechanical (or, hydro-mechanical) system is presented in this section, which has two capacitors connected in series.

Consider a fluid capacitor (Capacitance C_h) due to the gravity head of an incompressible liquid and another capacitor (Capacitance C_k) due to the flexibility of the container of an incompressible liquid, integrated together, as shown in Fig. 1.

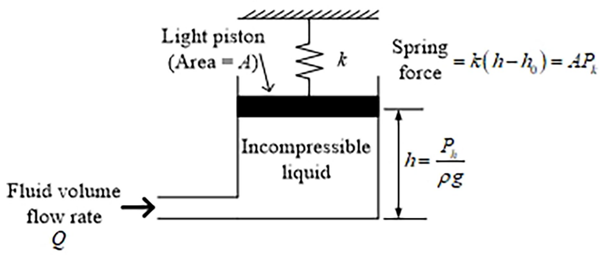


Fig. 1 Two series capacitors in a fluid-mechanical device.

Now it is shown that for this arrangement, the fluid volume flow rate Q is common and the pressures of the two capacitors (P_h and P_k) add. In the system, h = liquid height, and $P_h = \rho g h$, where ρ = mass density of the liquid. Also,

$$Q = A\frac{dh}{dt} = \frac{A}{\rho g}\frac{dP_h}{dt} \tag{1}$$

where, A = area of the flexible wall (spring-loaded piston). Hence,

$$C_h\frac{dP_h}{dt} = Q \tag{2}$$

with $C_h = \dfrac{A}{\rho g}$.

Next, spring force $= k(h - h_0)$, where h_0 = liquid height when the spring is relaxed. Hence, the pressure on the flexible wall, $P_k = \dfrac{k(h - h_0)}{A}$. This gives $\dfrac{dP_k}{dt} = \dfrac{k}{A}\dfrac{dh}{dt} = \dfrac{k}{A^2}\dfrac{dQ}{dt}$. Then, we have,

$$C_k\frac{dP_k}{dt} = Q \tag{3}$$

with $C_k = \dfrac{A^2}{k}$. From (2) and (3) it is clear that for the two capacitors, the volume flow rate Q is the same and the pressures P_h and P_k are additive. In particular, (2) and (3) can be written as, $\dfrac{dP_h}{dt} = \dfrac{1}{C_h}Q$ and $\dfrac{dP_k}{dt} = \dfrac{1}{C_k}Q$, which gives $\dfrac{d}{dt}(P_h + P_k) = \left(\dfrac{1}{C_h} + \dfrac{1}{C_k}\right)Q = \dfrac{1}{C_{eq}}Q$. Then, in the combined system, the overall pressure drop is $P_h + P_k$ with an equivalent fluid capacitance C_{eq} given by,

$$\frac{1}{C_{eq}} = \frac{1}{C_h} + \frac{1}{C_k} \tag{4}$$

This is the governing equation for the series connection of capacitors.

Note: The two capacitors (energy storage elements are not independent in this system, and two separate state variables should not be used to represent them. Specifically, a single state variable, for example, representing C_{eq}, should be used.

3 Redundant State Variables

In a systematic, integrated, and unified procedure for developing the state-space model of a multi-physics system, leading to a unique model, each energy storage element is assigned a state variable. If there are dependent energy-storage elements, as in the example of Section 2, that will resultant in redundant state variables and a non-minimal state-space realization. It follows that, dependent energy storage elements have to be identified and the corresponding redundant state variables have to be eliminated. The present section provides an example to illustrate this procedure.

Consider the multi-physics system consisting of both mechanical components and fluid components, shown in Fig. 2. A pump of pressure $P_s(t)$, which is a pressure source, pumps liquid into a uniform tank of area of cross-section A, through a pipe of circular cross-section. The pressure ripples in the liquid flow of the pipe are reduced before entering the tank, by means of an energy absorber consisting of a small fluid tank of area of cross-section a and a spring-loaded piston of mass m, stiffness k, and viscous damping constant b. Also given, I_f = fluid inertance in the pipe up to the energy absorber (passive pressure controller), R_f = fluid resistance in the pipe up to the energy absorber, and ρ = mass density of the liquid. First, an oriented linear graph is presented for the system. Using the linear graph, systematically, a complete state space model is developed for the system, with the following state variables: Q = volume flow rate of the liquid in the pipe before reaching the energy absorber, P_h = pressure difference of the liquid column in the energy absorber tank (not the pressure at the bottom of the liquid column), v = upward velocity of the piston, f_k = compressive force of the resisting spring attached to the piston, and P_H = pressure at the bottom of the main tank. Also, system output = H = liquid level in the main tank. For analytical convenience, the bulk modulus of the liquid and the flexibility of the pipes and the tanks are neglected.

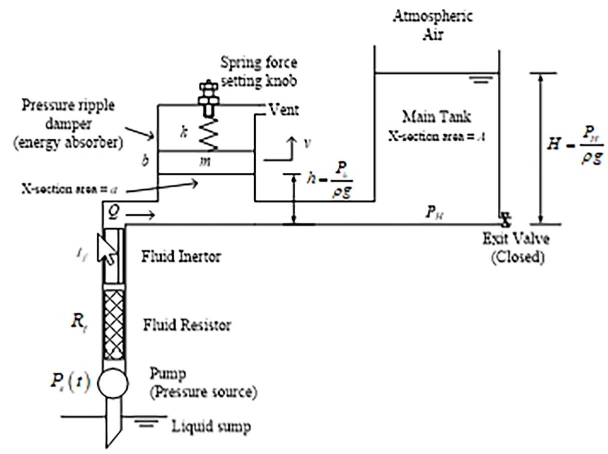


Fig. 2 A liquid pumping system.

The linear graph of the system is shown in Fig. 3. This is a "gyrator-coupled" system where the subsystem in the fluid domain is linked with the subsystem in the mechanical domain (the output domain) by a gyrator with parameter a, as shown.

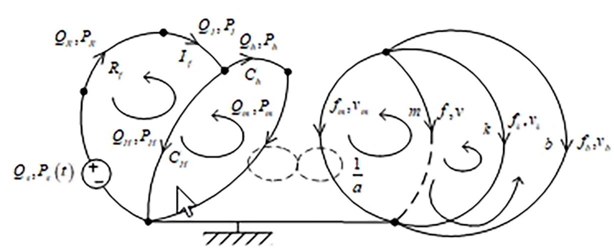


Fig. 3 Linear graph of the system.

The systematic procedure for developing the state-space model is presented now.

State-space Shell:

$C_H \dot{P}_H = Q_H$

$C_h \dot{P}_h = Q_h$

$I_f \dot{Q}_I = P_I$

$m\dot{v} = f$

$\dot{f}_k = kv_k$

Other Constitutive Equations:

$P_R = R_f Q_R$

$f_b = bv_b$

$$\left.\begin{array}{l} v_m = \dfrac{Q_m}{a} \\ f_m = -aP_m \end{array}\right\} \text{Fluid gyrator}$$

Node Equations:

$Q_s - Q_R = 0$(useless) ;

$Q_R - Q_I = 0$; $Q_I - Q_h - Q_H = 0$; $Q_h - Q_m = 0$; $-f_m - f - f_k - f_b = 0$

Loop Equations:

$P_s - P_H - P_I - P_R = 0$; $P_H - P_m - P_h = 0$; $v_m - v = 0$; $v - v_k = 0$; $v - v_b = 0$

Eliminate the auxiliary variables:

$Q_H = Q_I - Q_h = Q_I - Q_m = Q_I - av_m = Q_I - av$

$Q_h = Q_m = av_m = av$

$P_I = P_s - P_H - P_R = P_s - P_H - R_f Q_R = P_s - P_H - R_f Q_I f = -f_m - f_k - f_b = aP_m - f_k - bv_b = a(P_H - P_h) - f_k - bv$

$v_k = v$

Output $H = \dfrac{P_H}{\rho g}$

State vector $x = \begin{bmatrix} P_H & P_h & Q_I & v & f_k \end{bmatrix}^T$

Input vector $u = [P_s]$

Output vector $y = [H]$

State-space Model:

$\dot{x} = Ax + Bu$

$y = Cx + Du$

with,

$$A = \begin{bmatrix} 0 & 0 & \left(\dfrac{1}{C_H} = \dfrac{\rho g}{A}\right) & \left(-\dfrac{a}{C_H} = -\dfrac{a\rho g}{A}\right) & 0 \\ 0 & 0 & 0 & \left(\dfrac{a}{C_h} = \rho g\right) & 0 \\ -\dfrac{1}{I_f} & 0 & -\dfrac{R_f}{I_f} & 0 & 0 \\ \dfrac{a}{m} & -\dfrac{a}{m} & 0 & -\dfrac{b}{m} & -\dfrac{1}{m} \\ 0 & 0 & 0 & k & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{I_f} \\ 0 \\ 0 \end{bmatrix};\ C = \begin{bmatrix} \dfrac{1}{\rho g} & 0 & 0 & 0 & 0 \end{bmatrix};\ D = [0]$$

It is seen that the 2nd row of the system matrix (A) is directly proportional to the 5th row (through a constant multiplier). This tells us that the state variables x_2 and x_5 are not independent. Since the state-space model must contain the least number of state variables that can completely represent the dynamics of the system, it is necessary to eliminate one of these two state variables. There are many ways to perform tis elimination, but the most direct and simplest way is given now. Before doing this, note that the system order (which is equal to the order of the state vector) is equal to the number of "independent" energy storage elements in the system. Through physical examination (see Section 2) it should be clear that the fluid capacitor formed by the water column of the pressure damper (gravitational fluid capacitance) and the spring (of stiffness k) in the pressure damper are not independent. Since these two energy storage elements are not independent, two separate state variables should not be used to represent that subsystem, in the final state-space model. All these observations indicate that the state-space model (5th order) that was obtained before is not the correct final result, and it has to be reduced to a 4th order model. This is accomplished now.

The 2^{nd} and the 5^{th} state equations are:

$$\dot{x}_2 = \frac{a}{C_h} x_4$$

$$\dot{x}_5 = k x_4$$

with, $C_h = \frac{\rho g}{a}$. Hence, $x_5 = \frac{k}{\rho g} x_2$.

Note: The units of $\frac{k}{\rho g}$ are m^2, and hence the two state variables are physically compatible as well.

It follows that the proper state-space model should have only the first four state equations, with x_5 in the 4^{th} state equation replaced by $\frac{k}{\rho g} x_2$. This gives the proper (4^{th} order) state-space model is as follows.

Input $u = P_s(t)$; Output $H = \frac{P_H}{\rho g}$

State vector $x = [x_1\ x_2\ x_3\ x_4]^T = [P_H\ P_h\ Q_I\ v]^T$

Input vector $u = [u] = [P_s(t)]$

Output vector $y = [y] = \left[\frac{P_H}{\rho g}\right]$

State-space Model:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

The corresponding model matrices are:

$$A = \begin{bmatrix} 0 & 0 & \left(\frac{1}{C_H} = \frac{\rho g}{A}\right) & \left(-\frac{a}{C_H} = -\frac{a\rho g}{A}\right) \\ 0 & 0 & 0 & \left(\frac{a}{C_h} = \rho g\right) \\ -\frac{1}{I_f} & 0 & -\frac{R_f}{I_f} & 0 \\ \frac{a}{m} & -\frac{1}{m}(a + \frac{k}{\rho g}) & 0 & -\frac{b}{m} \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_f} \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} \frac{1}{\rho g} & 0 & 0 & 0 \end{bmatrix}; \quad D = [0]$$

4 Equivalent Transfer Function Model

System analysis in the frequency domain is known to be more convenient than in the time domain, mainly because the relevant operations are algebraic rather than differential. For achieving this objective, the state space model has to be converted in a transfer function, which can be conveniently accomplished through a transfer function linear graph (TF LG). An example of a multi-physics system similar (but not identical) to that in Section 3, is used to illustrate the pertinent procedure. The example system is fluid-mechanical, and in the LG, this coupling is represented by a gyrator, essentially giving a gyrator-coupled system [9].

The procedure of determining the time-domain input-output model (i.e., the input-output differential equation) from the state-space model, is illustrated as well. Then, the system transfer function is obtained conveniently. Another way to determine the system transfer function is to use the approach of transfer function linear graph (TF LG). In that procedure, the multi-domain system has to be converted into an equivalent single-domain TF LG. This procedure is also illustrated using the same example system. Among the two physical domains (fluid and mechanical) of the example system, the fluid domain is converted into an equivalent mechanical domain (the output domain of the considered system) before determining the transfer function of the system.

Consider the multi-physics system consisting of both mechanical components and fluid components, as shown in Fig. 4. A pump of pressure $P_s(t)$, which is a pressure source, pumps water into a uniform horizontal cylinder of area of cross-section A, which serves as the hydraulic actuator that drives a mechanical load. The combined mass of the actuator piston and the mechanical load is m, the resisting stiffness of the mechanical load is k, and the combined viscous damping constant of the actuator piston and the mechanical load is b. The water is pumped through a short pipe of circular cross-section. *Note*:

Assume that the water is incompressible. The pressure ripples in the water flow of the pipe are reduced before entering the actuator, by means of an energy absorber (hydraulic capacitor) consisting of a small fluid tank of area of cross-section A_c and a spring-loaded, light (massless) and smooth (no energy dissipation) piston with the resisting stiffness k_c.

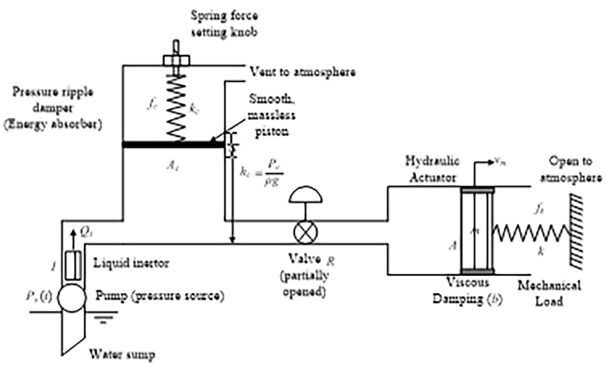


Fig. 4 Water pumping system for a hydraulic actuator.

Note: This stiffness is adjustable using a nut, as shown, but in this example, assume it to be a constant. The water flow into the actuator cylinder can be adjusted by means of a valve, as shown. It offers a fluid resistance R. Even though this is also adjustable, assume that it is a constant in the present example. This is the only notable fluid resistance that is present in the entire system (i.e., neglect any other hydraulic resistances). Given, I = fluid inertance in the pipe from the pump up to the energy absorber (passive pressure controller). Neglect any other fluid inertances. Also, ρ = mass density of the water, and g = acceleration due to gravity.

First, a complete linear graph for the system is presented. Using the linear graph, a complete state space model is developed for the system. The following state variables are used: v_m = velocity of the mechanical load (and also of the actuator piston), f_k = spring force of the mechanical load (attached to the actuator piston), f_c = compressive force of the spring of the energy absorber, Q_I = volume flow rate of the water in the pipe before reaching the energy absorber, and P_c = pressure difference of the water column (of height h_c) in the energy absorber tank (not the pressure at the bottom of this water column). Note: $P_c = \rho g h_c$. System output = v_m = velocity of the load (also, of the actuator piston). In this example, the bulk modulus of water and the flexibility of the pipes, the actuator cylinder and the absorber tank. are neglected.

From the state-space model, the input-output differential equation (input = $P_s(t)$, output = v_m) is obtained. From that equation, the system transfer function is obtained. Next, the TF LG corresponding to the LG is obtained. Then it is reduced into a TF LG that is entirely in the mechanical domain. From that, systematically the system transfer function is obtained. It is shown that his result is identical to that obtained before.

The linear graph of the system is shown in Fig. 5(a).

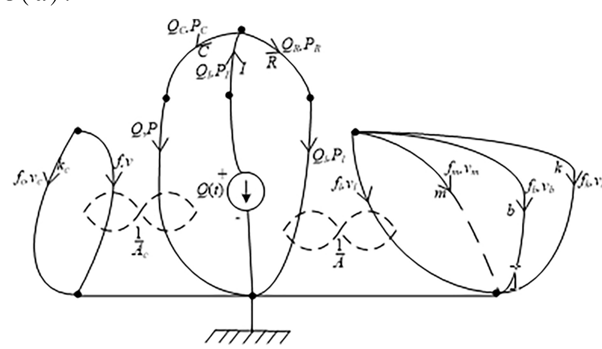


Fig. 5(a) Linear graph of the system.

The state-space model is developed now.

State-space Shell:

$$m\dot{v}_m = f_m$$

$$\dot{f}_k = kv_k$$

$$\dot{f}_c = k_c v_c$$

$$I\dot{Q}_I = P_I$$

$$C\dot{P}_C = Q_C$$

Note: $C = \dfrac{A_c}{\rho g}$

Other Constitutive Equations:

$f_b = bv_b$

$P_R = RQ_R$

$$\left.\begin{aligned} v_l &= \frac{Q_l}{A} \\ f_l &= -AP_l \end{aligned}\right\} \text{Fluid gyrator } A$$

$$\left.\begin{aligned} v &= \frac{Q}{A_c} \\ f &= -A_c P \end{aligned}\right\} \text{Fluid gyrator } A_c$$

Node Equations:

$Q_s - Q_I = 0$(useless) ; $Q_I - Q_R - Q_C = 0$;

$Q_C - Q = 0$;

$-f - f_c = 0$; $Q_R - Q_l = 0$; $-f_l - f_m - f_b - f_k = 0$

Loop Equations:

$-P_s(t) + P_I + P_C + P = 0$;

$-P_l - P_R + P_C + P = 0$;

$-v + v_c = 0$; $-v_m + v_l = 0$; $-v_b + v_m = 0$; $-v_k + v_m = 0$

Eliminate auxiliary variables:

$$f_m = -f_l - f_b - f_k = AP_l - bv_b - f_k$$
$$= A(-P_R + P_C + P) - bv_m - f_k$$
$$= A\left(-RQ_R + P_C - \frac{1}{A_c}f\right) - bv_m - f_k$$
$$= A\left(-RQ_l + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k$$
$$= A\left(-RAv_l + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k$$
$$= A\left(-RAv_m + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k$$
$$= -(A^2R + b)v_m - f_k + \frac{A}{A_c}f_c + AP_C$$

$$kv_k = kv_m$$

$$k_c v_c = k_c v = \frac{k_c}{A_c}Q = \frac{k_c}{A_c}Q_C = \frac{k_c}{A_c}(Q_I - Q_R)$$
$$= \frac{k_c}{A_c}(Q_I - Av_m) \text{ [from above result}$$
$$= -\frac{k_c A}{A_c}v_m + \frac{k_c}{A_c}Q_I$$

$$P_I = -P - P_C + P_s(t) = \frac{1}{A_c}f - P_C + P_s(t)$$
$$= -\frac{1}{A_c}f_c - P_C + P_s(t)$$

$Q_C = Q_I - Q_R = Q_I - Av_m$[from above

Output $= v_m$

Vector-matrix form:

State vector $x = [v_m \quad f_k \quad f_c \quad Q_I \quad P_C]^T$

Input vector $u = [u] = [P_s(t)]$

Output vector $y = [y] = [v_m]$

State-space Model:

$\dot{x} = Ax + Bu$

$y = Cx + Du$

with

$$A = \begin{bmatrix} -\frac{(A^2R+b)}{m} & -\frac{1}{m} & \frac{A}{mA_c} & 0 & \frac{A}{m} \\ k & 0 & 0 & 0 & 0 \\ -\frac{k_cA}{A_c} & 0 & 0 & \frac{k_c}{A_c} & 0 \\ 0 & 0 & -\frac{1}{IA_c} & 0 & -\frac{1}{I} \\ -\frac{A}{C} & 0 & 0 & \frac{1}{C} & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I} \\ 0 \end{bmatrix}; C = [1 \quad 0 \quad 0 \quad 0 \quad 0]; D = [0]$$

From the system matrix (A) it is clear that its 3rd row is directly proportional to the 5th row (through a constant multiplier). This tells that the state variables x_3 and x_5 are not independent. Since the state-space model must contain the least number of state variables that completely represent the dynamics of the system, it is necessary to eliminate one of these two state variables. As in Section 3, through physical examination it should be clear that the fluid capacitor formed by the water column of the pressure damper (gravitational fluid capacitance) and the spring (of stiffness k_c) in the pressure damper are not independent. Hence, the state-space model (5^{th} order) that was obtained before, has to be reduced to a 4^{th} order model. This is accomplished now.

From the gyrator equations (with A_c), we have,

$$A_c P = -f$$

Differentiate and substitute some previous equations:

$$A_c \dot{P} = -\dot{f} = \dot{f}_c = k_c v_c = k_c v = k_c \frac{Q}{A_c}$$

We have: $\frac{A_c^2}{k_c}\dot{P} = Q$

This result tells us that the equivalent fluid capacitance of the spring element in the pressure damper is: $C_c = \frac{A_c^2}{k_c}$. From Fig. 5(a) it is clear that this capacitor is in "series" with the actual fluid capacitor (of capacitance C (with common flow and additive pressure difference). The combined, equivalent fluid capacitance C_e of these two series elements is given by, $\frac{1}{C_e} = \frac{1}{C} + \frac{1}{C_c} = \frac{\rho g}{A_c} + \frac{k_c}{A_c^2}$. Hence,

$$C_e = \frac{A_c^2}{\rho g A_c + k_c} \tag{5}$$

The corresponding linear graph is shown in Fig. 5(b).

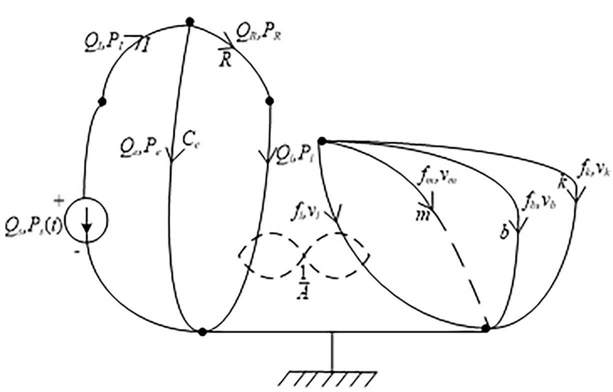


Fig. 5(b) LG of the equivalent system with independent energy storage elements.

Now, the state-space model corresponding to this linear graph is obtained, which is the correct state-space model.

State-space Shell:

$$m\dot{v}_m = f_m$$

$$\dot{f}_k = k v_k$$

$$I\dot{Q}_I = P_I$$

$$C_e \dot{P}_e = Q_e$$

Note: $C_e = \frac{A_c^2}{\rho g A_c + k_c}$. The variables P_e and Q_e are as in Fig. 5(b).

Other Constitutive Equations:

$$f_b = b v_b$$

$$P_R = R Q_R$$

$$\left.\begin{aligned} v_l &= \frac{Q_l}{A} \\ f_l &= -AP_l \end{aligned}\right\} \text{Fluid gyrator } A$$

Node Equations:

$Q_s - Q_I = 0(\text{useless})$; $Q_I - Q_R - Q_e = 0$;

$Q_R - Q_l = 0$; $-f_l - f_m - f_b - f_k = 0$

Loop Equations:

$-P_s(t) + P_I + P_e = 0$; $-P_l - P_R + P_e = 0$;

$-v_m + v_l = 0$; $-v_b + v_m = 0$; $-v_k + v_m = 0$

Eliminate auxiliary variables:

$$\begin{aligned} f_m &= -f_l - f_b - f_k = AP_l - bv_b - f_k \\ &= A(-P_R + P_e) - bv_m - f \\ &= A(-RQ_R + P_e) - bv_m - f_k \\ &= A\left(-RQ_l + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k \\ &= A\left(-RAv_l + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k \\ &= A\left(-RAv_m + P_C + \frac{1}{A_c}f_c\right) - bv_m - f_k \\ &= -(A^2R + b)v_m - f_k + \frac{A}{A_c}f_c + AP_C \end{aligned}$$

$$kv_k = kv_m$$

$$P_I = -P_e + P_s(t)$$

$Q_e = Q_I - Q_R = Q_I - Av_m$[as before

Output $= v_m$

Vector-matrix form of the state-space model:

State vector $x = [x_1\ x_2\ x_3\ x_4]^T = [v_m\ f_k\ Q_I\ P_e]^T$

Input vector $u = [u] = [P_s(t)]$

Output vector $y = [y] = [v_m]$

State-space Model:

$\dot{x} = Ax + Bu$

$y = Cx + Du$

The corresponding model matrices are:

$$A = \begin{bmatrix} -\dfrac{(A^2R+b)}{m} & -\dfrac{1}{m} & 0 & \dfrac{A}{m} \\ k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{I} \\ -\dfrac{A}{C_e} & 0 & \dfrac{1}{C_e} & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{I} \\ 0 \end{bmatrix};\ C = [1 \quad 0 \quad 0 \quad 0];\ D = [0]$$

Now, the input-output differential equation corresponding to the state-space model, is determined. The state equations may be written as:

$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{14}x_4$

$\dot{x}_2 = a_{21}x_1$

$\dot{x}_3 = a_{34}x_4 + b_3 u$

$\dot{x}_4 = a_{41}x_1 + a_{43}x_3$

with,

$a_{11} = -\dfrac{(A^2R+b)}{m};\ a_{12} = -\dfrac{1}{m};\ a_{14} = \dfrac{A}{m};$

$a_{21} = k;\ a_{34} = -\dfrac{1}{I};\ a_{41} = -\dfrac{A}{C_e};\ a_{43} = \dfrac{1}{C_e};\ b_3 = \dfrac{1}{I}$ and $C_e = \dfrac{A_c^2}{\rho g A_c + k_c}$.

The state equations may be further rewritten as:

$$\dot{y} = a_{11}y + a_{12}x_2 + a_{14}x_4 \tag{6}$$

$$\dot{x}_2 = a_{21}y \tag{7}$$

$$\dot{x}_3 = a_{34}x_4 + b_3 u \tag{8}$$

$$\dot{x}_4 = a_{41}y + a_{43}x_3 \tag{9}$$

We have to express these 4 equations as a single equation in u and y only. This done as follows:

$$\frac{d(6)}{dt}:\ \ddot{y} = a_{11}\dot{y} + a_{12}a_{21}y + a_{14}(a_{41}y + a_{43}x_3)$$

$$= (a_{12}a_{21} + a_{14}a_{41})y + a_{11}\dot{y} + a_{14}a_{43}x_3 \tag{10}$$

$$= a_1 y + a_{11}\dot{y} + a_2 x_3$$

$$a_1 = a_{12}a_{21} + a_{14}a_{41} = -\frac{1}{m}k + \frac{A}{m}\left(-\frac{A}{C_e}\right)$$

where, $= -\dfrac{k}{m} - \dfrac{A^2}{mC_e}$;

$$a_2 = a_{14}a_{43} = \frac{A}{m}\frac{1}{C_e} = \frac{A}{mC_e}$$

$$\frac{d(8)}{dt}:\ \ddot{x}_3 = a_{34}(a_{41}y + a_{43}x_3) + b_3\dot{u}$$

Substitute (10):

$$\frac{1}{a_2}(\ddddot{y} - a_1\ddot{y} - a_{11}\dddot{y}) = a_{34}a_{41}y + \frac{a_{34}a_{43}}{a_2}(\ddot{y} - a_1 y - a_{11}\dot{y}) + b_3\dot{u}$$

Rearrange:

$$\ddddot{y} - a_{11}\dddot{y} - (a_1 + a_{34}a_{43})\ddot{y} + a_{11}a_{34}a_{43}\dot{y} + (a_1a_{34}a_{43} - a_2a_{34}a_{41})y = a_2b_3\dot{u}$$

Also, we have:

$$a_1 + a_{34}a_{43} = -\frac{k}{m} - \frac{A^2}{mC_e} + \left(-\frac{1}{I}\right)\frac{1}{C_e} = -\frac{k}{m} - \frac{A^2}{mC_e} - \frac{1}{IC_e}$$

$$a_{11}a_{34}a_{43} = -\frac{(A^2R+b)}{m}\left(-\frac{1}{IC_e}\right) = \frac{(A^2R+b)}{mIC_e}$$

$$a_1a_{34}a_{43} - a_2a_{34}a_{41} = \left(-\frac{k}{m} - \frac{A^2}{mC_e}\right)\left(-\frac{1}{IC_e}\right) - \frac{A}{mC_e}\left(-\frac{1}{I}\right)\left(-\frac{A}{C_e}\right) = \frac{1}{IC_e}\left(\frac{k}{m} + \frac{A^2}{mC_e} - \frac{A^2}{mC_e}\right) = \frac{k}{mIC_e}$$

$$a_2b_3 = \frac{A}{mC_e}\frac{1}{I} = \frac{A}{mC_eI}$$

Note: At this stage, it is a good idea to check the physical units of each of these terms and verify that they are consistent. If not, that means, the result has errors.

Substitute the expressions for the coefficients:

$$\frac{d^4y}{dt^4}+\frac{(A^2R+b)}{m}\frac{d^3y}{dt^3}+\left(\frac{k}{m}+\frac{A^2}{mC_e}+\frac{1}{IC_e}\right)\frac{d^2y}{dt^2}+\frac{(A^2R+b)}{mIC_e}\frac{dy}{dt}+\frac{k}{mIC_e}y=\frac{A}{mC_eI}\frac{du}{dt}.$$

This gives the input-output differential equation,

$$mIC_e\frac{d^4y}{dt^4}+IC_e(A^2R+b)\frac{d^3y}{dt^3}+(kIC_e+A^2I+m)\frac{d^2y}{dt^2}+(A^2R+b)\frac{dy}{dt}+ky=A\frac{du}{dt}$$

As expected, a 4th order input-output differential equation model is obtained for this 4th order system (having 4 independent energy storage elements).

The corresponding system transfer function is obtained by simply substituting the time-derivative operation by the Laplace variable s:

$$\frac{V_m(s)}{P_s(s)}=\frac{As}{mIC_es^4+IC_e(A^2R+b)s^3+(kIC_e+A^2I+m)s^2+(A^2R+b)s+k}$$

This result is verified by using the method of TF LG, next.

The transfer-function linear graph (TF LG) of the system in Fig. 5(b) is shown in Fig. 5(c).

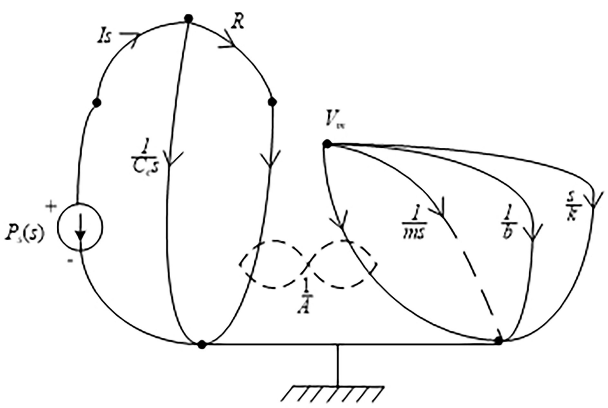


Fig. 5(c) The Transfer-function linear graph of the fluid-mechanical system. The fluid domain is now converted into the Thevenin form [9], as shown in Fig. 5(d).

Further reduction of this LG is not required. By following the usual procedure for Thevenin circuit development, we have,

Equivalent (open-circuit) pressure source

$$P_{oc}(s)=\frac{P_s(s)}{(Is+\frac{1}{C_es})}\times\frac{1}{C_es}=\frac{P_s(s)}{(IC_es^2+1)} \tag{11}$$

The potential divider method is used in writing this equation.

Equivalent fluid impedance

$$Z_e=\frac{Is\times\frac{1}{C_es}}{(Is+\frac{1}{C_es})}+R=\frac{Is}{(IC_es^2+1)}+R \tag{12}$$

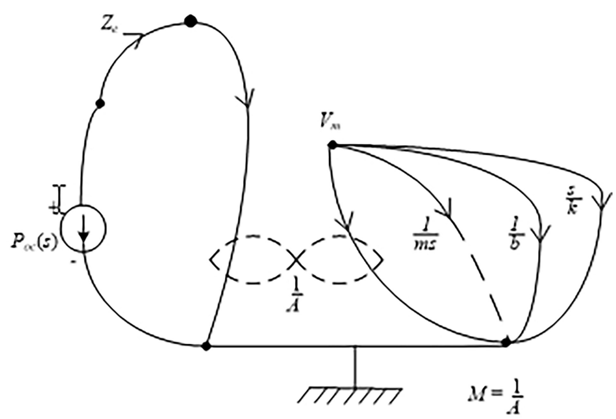
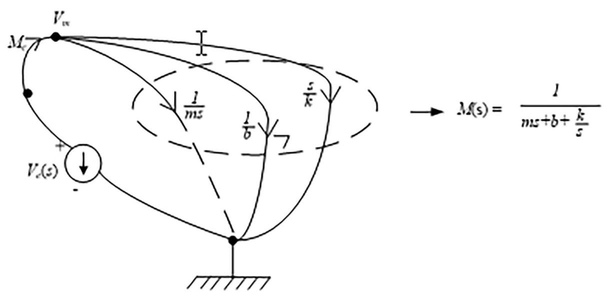


Fig. 5(d) The TF LG with the fluid domain in the Thevenin form.

Here, after killing (i.e., shorting) the fluid source, the resulting two parallel branches have been combined and then the series branch is added.

Next, the fluid domain is converted into an equivalent mechanical domain, through the gyrator (this is a gyrator-couple fluid-mechanical system). The equivalent TF LG shown in Fig. 5(e) is obtained. This equivalent system is entirely in the mechanical domain.

The domain conversion of the gyrator-coupled



two-domain system is carried using the standard re-

Fig. 5(e) The equivalent TF LG entirely in the mechanical domain.

sult of converting a Thevenin segment in one domain into an equivalent Thevenin segment in the other domain [9].

In the converted segment, the equivalent velocity source is: $V_e(s) = \frac{M}{Z_e}P_{oc}(s)$ and the

series mobility in that segment is: $M_e = \frac{M^2}{Z_e}$

In the present problem, $M = \frac{1}{A}$

Substitute (11) and (v12):

$$V_e(s) = \frac{M}{Z_e}P_{oc}(s) = \frac{P_s(s)}{A(IC_es^2+1)\times\left[\frac{Is}{(IC_es^2+1)}+R\right]} = \frac{P_s(s)}{A[Is+R(IC_es^2+1)]}$$

$$M_e = \frac{M^2}{Z_e} = \frac{1}{A^2\left[\frac{Is}{(IC_es^2+1)}+R\right]} = \frac{(IC_es^2+1)}{A^2[Is+R(IC_es^2+1)]}$$

In Fig. 5(e), the three parallel branches of m, b, and k can be combined to give the mobility,

$$M(s) = \frac{1}{ms+b+k/s}$$

Applying potential division to Fig. 5(e), we get

$$V_m = \frac{M}{M_e+M}\times V_e(s) = \frac{V_e(s)}{M_e/M+1} =$$

$$\frac{P_s(s)}{A[Is+R(IC_es^2+1)]}\times\frac{1}{\left(\frac{(IC_es^2+1)(ms+b+k/s)}{A^2[Is+R(IC_es^2+1)]}+1\right)} =$$

$$\frac{AP_s(s)}{(IC_es^2+1)(ms+b+k/s)+A^2[Is+R(IC_es^2+1)]} =$$

$$\frac{AsP_s(s)}{(IC_es^2+1)(ms^2+bs+k)+A^2s[Is+R(IC_es^2+1)]} =$$

$$\frac{AsP_s(s)}{mIC_es^4+(IC_eb+A^2RIC_e)s^3+(m+kIC_e+A^2I)s^2+(b+A^2R)s+k}$$

Hence, the system transfer function is

$$\frac{V_m}{P_s(s)} = \frac{As}{mIC_es^4+(IC_eb+A^2RIC_e)s^3+(m+kIC_e+A^2I)s^2+(b+A^2R)s+k}$$

This transfer function is identical to what was obtained previously, by the time-domain approach.

5 Conclusions

The present paper concerned the use of linear graphs to generate a minimal model for a multi-physics system. The number of state variables used in a state-space model should be the minimum number that can completely represent the dynamic state of the system. The paper presented an approach to recognize redundant state variables, and through that obtain a minimal state-space model. Initially, state variables were assigned to all the energy-storage elements of the system. Next, a way to recognize the dependent energy storage elements and thereby remove the redundant state variables was presented. System analysis in the frequency domain is known to be more convenient than in the time domain, mainly because the relevant operations are algebraic rather than differential. For achieving this objective, the system transfer function is needed. The direct way to convert a state space model into the corresponding input-output differential equation, and thereby obtain the system transfer function was presented. It was shown that the same result could be obtained through the transfer function linear graph (TF LG) of the system. In a multi-physics system, first the physical domains have to be converted into an equivalent single domain (preferably, the output domain of the system), when using the method of TFLG. This procedure was illustrated as well, in the present paper.

Acknowledgment

This work has been supported by research grants from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

References

[1] Simidjievski, N., Todorovski, L., and Dzeroski, S. (2015). "Predicting long-term population dynamics with bagging and boosting of process-based models," *Expert Systems with Applications*, Vol. 42, No. 22, pp. 8484 – 8496.

[2] Groff, J. (2013). "Dynamic Systems Modeling in Educational System Design and Policy," *Journal of New Approaches in Educational Research*, Vol. 2, No. 2, pp. 72-81.

[3] Iskander, J., Hossny, M., Nahavand, S., and del Porto, L. (2018). "An ocular biomechanic model for dynamic simulation of different eye movements," *Journal of Biomechanics*, Vol. 71, pp. 209-216.

[4] Li, P., Li, Y., and Seem, J.E. (2010). "Dynamic Modeling and Stability Analysis of Water-Cooled Centrifugal Chillers," *Proc. ASME Dynamic Systems and Control Conference*, Volume 2, Cambridge, MA, pp. 703-710.

[5] Lang, H., Wang, Y., and de Silva, C.W. (2017). "Semi-empirical Modeling and Parameter Analysis for the Development of a High Speed Resistance-welding Process," *International Journal of Modelling and Simulation*, pp. 1-9.

[6] Daryabak, M., Filizadeh, S., Jatskevich, J., et al. (2014). "Modeling of LCC-HVDC Systems using Dynamic Phasors," *IEEE Trans. on Power Delivery*, Vol. 29, No. 4, pp. 1989 – 1998.

[7] Thomas, M. and Tesar, D. (1982). "Dynamic Modeling of Serial Manipulator Arms," *Journal of Dynamic Systems Measurement and Control*, *Trans. ASME*, Vol. 104, No. 3, pp. 218-228.

[8] De Silva, C.W. (2012). "Linear-graph Modeling Paradigms for Mechatronic Systems," *The International Journal of Mechanical Engineering Education*, Vol. 40, No. 4, pp. 305-328.

[9] De Silva, C.W. (2018). MODELING OF DYNAMIC SYSTEMS—With Engineering Applications, Taylor & Francis/CRC Press, Boca Raton, FL.

[10] Bahill, A.T. and Szidarovszky, F. (2009). "Comparison of Dynamic System Modeling Methods," *Systems Engineering*, Vol 12, No. 3, pp. 184-200.

[11] Calvanese, D., Montali, M., Patrizi, F., and De Giacomo, G. (2015). "Description Logic Based Dynamic Systems: Modeling, Verification, and Synthesis," *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence* (*IJCAI*), Buenos Aires, Argentina, pp. 4247-4253.

[12] Ledin, J. (2000). "Modeling Dynamic Systems," *Embedded Systems Programming*, pp. 84-96, August.

[13] Stein, J.L. and Louca, L.S. (2010). "Physical System Modeling," *Journal of Dynamic Systems Measurement and Control*, *Trans. ASME*, Vol. 132, No. 6, pp. 1-2.

[14] De Silva, C.W. and Pourazadi, S. (2013). "Some Generalizations of Linear-graph Modeling for Dynamic Systems," *International Journal of Control*, Vol. 86, No. 11, pp. 1990-2005.

[15] De Silva, C.W. (2019). "A Systematic Approach for Modelling Multi-physics Systems," The International Journal of Mechanical Engineering Education, https://doi.org/10.1177/0306419019868801.

Authors' Biographies

Clarence W. DE SILVA, received the Ph.D. degrees from the Massachusetts Institute of Technology, Cambridge, MA, USA, in 1978, and from the University of Cambridge, Cambridge, U.K., in 1998, the honorary D.Eng. degree from the University of Waterloo, Waterloo, ON, Canada, in 2008, and the Higher Doctorate, Sc.D., from the University of Cambridge in 2020. He is a Professor of mechanical engineering at The University of British Columbia, Vancouver, BC, Canada. He is a fellow of: IEEE, ASME, Canadian Academy of Engineering, and Royal Society of Canada. His appointments include the Tier 1 Canada Research Chair in Mechatronics and Industrial Automation, Professorial Fellow, Peter Wall Scholar, Mobil Endowed Chair Professor, and NSERCBC Packers Chair in Industrial Automation.
Email: desilva@mech.ubc.ca